Bounds on \((n, r)\)-arcs and their application to linear codes

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Abstract

This article reviews some of the principal and recently-discovered lower and upper bounds on the maximum size of \((n, r)\)-arcs in \(\text{PG}(2, q)\), sets of \(n\) points with at most \(r\) points on a line. Some of the upper bounds are used to improve the Griesmer bound for linear codes in certain cases. Also, a table is included showing the current best upper and lower bounds for \(q \leq 19\), and a number of open problems are discussed.

1 Background

The weight of a vector \(v\) is the number of non-zero coordinates of \(v\). Let \(V\) be the \(n\)-dimensional vector space over \(\mathbb{F}_q\). A linear \([n, k, d]\)-code \(C\) over \(\mathbb{F}_q\) is a \(k\)-dimensional subspace of \(V\) all of whose non-zero vectors have weight at least \(d\). Let \(v_1, v_2, \ldots, v_k\) be a basis for \(C\) and for \(i = 1, 2, \ldots, n\) define vectors \(u_i\) of \(V\), by the rule

\[
(u_i)_j = (v_j)_i.
\]

In other words, the \(j\)-th co-ordinate of \(u_i\) is the \(i\)-th coordinate of \(v_j\). For all \(a \in (\mathbb{F}_q)^k\) the vector \(\sum_{j=1}^{k} a_j v_j\) has at most \(n - d\) zero coordinates and so, for \(i = 1, 2, \ldots, n\),

\[
\sum_{j=1}^{k} a_j (v_j)_i = 0
\]

has at most \(n - d\) solutions. Hence

\[
\sum_{j=1}^{k} a_j (u_i)_j = 0
\]

has at most \(n - d\) solutions, or in other words there are at most \(n - d\) of the \(n\) vectors \(u_i\) on the hyperplane with equation

\[
\sum_{j=1}^{k} a_j X_j = 0.
\]

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The matrix whose rows are the vectors $v_i$, or equivalently whose columns are the vectors $u_i$, is called a generator matrix of the code $C$. An $(n, r)$-arc in $\text{PG}(k-1, q)$ is a set of $n$ points $K$ with the property that every hyperplane is incident with at most $r$ points of $K$ and there is some hyperplane incident with exactly $r$ points of $K$. Hence an $(n, n-d)$-arc in $\text{PG}(k-1, q)$ is equivalent to a linear $[n, k, d]$-code where $\langle u_i \rangle \neq \langle u_m \rangle$ for $i \neq m$, that is, linear codes for which any two columns of the generator matrix are linearly independent.

The aim of this article is to formulate the bounds on $(n, r)$-arcs as bounds that will look more familiar to coding theorists, to survey recent improvements and list a number of open problems.

For further background to linear codes see [43] or [36], and for $(n, r)$-arcs in $\text{PG}(2, q)$, see [32, Chapter 12]. In various articles and books, when $r$ is large, the complement of a $(n, r)$-arc is considered; this is called a $t$-fold blocking set.

A $t$-fold blocking set with respect to hyperplanes is a set of points that is incident with at least $t$ points of every hyperplane and there is some hyperplane incident with exactly $t$ points of the set. In this article it is preferred to leave everything in the language of $(n, r)$-arcs. Note that an $(n, r)$-arc in $\text{PG}(k-1, q)$ is the complement of a $(q+1-r)$-fold blocking set of hyperplanes of size $q^{k-1} + q^{k-2} + \ldots + 1 - n$. An alternative description used is that of an $\{n, m; N, q\}$-minihyper which is an $m$-fold blocking set with respect to hyperplanes of $\text{PG}(N, q)$ of size $n$.

## 2 Bounds on $(n, r)$-arcs

In this section, attention is restricted to the case $k = 3$; that is, $(n, r)$-arcs in the plane $\text{PG}(2, q)$ are considered.

Let $K$ be an $(n, r)$-arc and $P$ be a point of $K$. Each line incident with $P$ contains at most $r - 1$ points of $K \backslash P$ and the trivial upper bound is obtained:

$$n \leq (r - 1)(q + 1) + 1 = (r - 1)q + r.$$  

Cossu [21] noted that when the upper bound is attained every line is incident with either zero or $r$ points of $K$ and if $r \leq q$, by counting points of $K$ on lines incident with a point $Q$ not in $K$, that $r$ divides $q$. In the case when $q$ is even, such arcs exist for every $r$ dividing $q$. In the cases $r = 2$ and $r = q/2$, the arcs are called hyperovals and dual hyperovals respectively; the known examples are detailed in [34]. There are examples for all $r$ dividing $q$, due to Denniston. Recently, Mathon [41], Mathon and Hamilton [29] and Hamilton [28] constructed many new examples. In the case that $q$ is odd the upper bound can be realised only in the trivial cases $r = 1$, $r = q$ and $r = q + 1$. This was shown in [7]; see [6] for an easier proof. The investigation of $(n, r)$-arcs was initiated by Barlotti [8] whose early work now implies that, if $(r, q) \neq (2^i, 2^h)$ and $2 < r < q$, then

$$n \leq (r - 1)q + r - 2.$$  

An almost complete table of the known upper bounds can be found in [34, Table 5.2]. The only bound to have been published since then is in the case $r \mid q$ and $q$ odd, where Weiner [46] improved Szönyi’s bound [42],

$$n \leq (r - 1)q + r - \frac{1}{2} \sqrt{q},$$  

2
to

\[ n \leq (r - 1)q + r - \frac{1}{4}\sqrt{q} \]

for \( r \leq \frac{1}{2}\sqrt{q} \). There is one bound that appears in [34, Table 5.2] which is attributed to an unpublished manuscript of the first author. However, the bound is not quite correct as the strictly less than should be a less than or equal to. It is obtained as a corollary to the following theorem, to which a proof is provided since it has not appeared anywhere else.

**Theorem 2.1.** If there exists an \(((r - 1)q + \epsilon, r)\)-arc \( K \) with \( \epsilon \geq 1 \) in a projective plane \( \pi \) of order \( q \) which has no skew line, then

\[ r^2 - \epsilon q + \epsilon(\epsilon - r) - r \geq 0. \]

**Proof** Let \( n = (r - 1)q + \epsilon \). Counting points of \( K \) on each line through a point \( P \) of \( K \), it is seen that every line meets \( K \) in at least \( \epsilon \) points. Bruen’s idea [17] is extended to look at the inequality,

\[ \sum_{i=\epsilon}^r (r - i)(i - \epsilon)\tau_i \geq 0, \]

where \( \tau_i \) is the number of lines meeting \( K \) in \( i \) points. Standard counting arguments for a point set in a projective plane give

\[ \sum_{i=\epsilon}^r \tau_i = q^2 + q + 1, \quad \sum_{i=\epsilon}^r i\tau_i = n(q + 1), \quad \sum_{i=\epsilon}^r i(i - 1)\tau_i = n(n - 1), \]

and, combining these with the inequality, implies that

\[ -n(n - 1) + (\epsilon + r - 1)n(q + 1) - \epsilon r(q^2 + q + 1) \geq 0. \]

By calculation this gives

\[ r^2 - \epsilon q + \epsilon(\epsilon - r) - r \geq 0. \]

\[ \square \]

**Corollary 2.2.** An \((n, r)\)-arc \( K \) in a projective plane \( \pi \) of order \( q \) which has no skew line satisfies

\[ n \leq (r - 1)q + \left\lfloor \frac{r^2}{q} \right\rfloor, \]

and if \( \sqrt{q} \) divides \( r \) then

\[ n < (r - 1)q + \left\lfloor \frac{r^2}{q} \right\rfloor. \]

**Proof** Theorem 2.1 provides a contradiction for \( \epsilon \geq r^2/q \).

Corollary 2.2 in combination with the following from [4] can always be used to provide an upper bound.

An \((n, r)\)-arc \( K \) in a projective plane \( \text{PG}(2, q) \) which has a skew line satisfies

\[ n \leq (r - 1)q + p^\epsilon, \]

where \( p \) is the order of \( \pi \).
where \((r, q) = p^e\).

Table A lists all \(r\) for which there is known to exist an \((n, r)\)-arc in \(\text{PG}(2, q)\) and the maximum value of \(n\) known in that case. An \((n, r)\)-arc in \(\text{PG}(2, q)\), with \(n > (r - 2)q + r\), is equivalent to a code meeting the Griesmer bound, see Section 3.

In the table, the integer \(q = p^h\) is exceptional if \(h\) is odd, \(h \geq 3\) and \(p\) divides \([2\sqrt{q}]\).

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<td>(2^h)</td>
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<td>((r - 2)q + 1 + [2\sqrt{q}])</td>
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<td>(q) exceptional</td>
<td>((r - 2)q + [2\sqrt{q}])</td>
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<td>(q) odd</td>
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<td>(q = p^h)</td>
<td>((r - 1)q + q - r)</td>
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<tr>
<td>(r \geq q + 2 - \sqrt{q})</td>
<td>(q) square</td>
<td>((r - 1)q + r - \sqrt{q}(q + 1 - r))</td>
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<td>(q) even</td>
<td>((r - 1)q + 2)</td>
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<tr>
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<tr>
<td>(r = q - 1)</td>
<td>(q)</td>
<td>((r - 1)q + 1)</td>
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</table>

Table A: The known families of \((n, r)\)-arcs in \(\text{PG}(2, q)\) with \(n > (r - 2)q + r\)

Large arcs can also be constructed from the set of rational points of an algebraic curve, sometimes by adding extra points; see Daskalov and Jiménez Contreras [24], Giulietti et. al. [27] and Voloch [44].

3 Bounds on linear codes

In this section, some of the upper bounds on \((n, r)\)-arcs are reformulated in terms of linear codes. This gives a Griesmer-like bound (3.1) for three-dimensional codes which is essentially nothing new but only novel in its formulation. Corollary 3.2 generalises the bound to higher-dimensional codes.

Recall that for a linear \([n, k, d]\)-code the Griesmer bound, [43, Theorem 5.2.6], states that

\[
n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil.
\]

An \((n, r)\)-arc in \(\text{PG}(2, q)\) is equivalent to a linear \([n, 3, n - r]\)-code and so the Griesmer bound tells us, assuming \(d \leq q^2\),

\[
n \geq \sum_{i=0}^{2} \left\lceil \frac{n - r}{q^i} \right\rceil = n - r + \left\lceil \frac{n - r}{q} \right\rceil + 1.
\]

Hence we have the upper bound

\[
n \leq (r - 1)q + r
\]
and equality in the Griesmer bound if and only if \( n > (r - 2)q + r \).

An \((n, r)\)-arc in \(PG(2, q)\) satisfies the upper bound,

\[
n \leq (r - 1)q + 1,
\]

in the cases when

(i) (Blokhuis, see [2]) \( q \) is prime and \( r \leq (q + 3)/2 \);

(ii) (Section 2) \((r, q) = 1\) and \( r < \sqrt{2q} + 1 \);

(iii) (Blokhuis [12]) \((r, q) = 1\) and there is a line skew from the \((n, r)\)-arc;

(iv) (Weiner [46]) \( q \) is odd, \( r \mid q \) and \( r < \frac{1}{4}\sqrt{q} \).

The following theorem reformulates this bound in terms of linear codes. Put \( q = p^h \), where \( p \) is prime.

**Theorem 3.1.** Suppose that one of the following holds:

(i) \( q \) is prime and \( d \leq (q - 1)(q + 3)/2 \);

(ii) \( \lceil d/(q - 1) \rceil \neq -1 \pmod{p} \) and \( d \leq \sqrt{2q}(q - 1) \);

(iii) \( \lceil d/(q - 1) \rceil \neq -1 \pmod{p} \), \( d \leq q^2 \) and there is a codeword of weight \( n \);

(iv) \( q \) is odd, \( n = d + p^e \) for some \( e \) and \( d < (\frac{1}{4}\sqrt{q} - 1)(q - 1) \).

Then a linear \([n, 3, d]\) code over \( F_q \) satisfies

\[
n \geq d + \left\lfloor \frac{d}{q - 1} \right\rfloor + \left\lfloor \frac{d}{q^2} \right\rfloor.
\]

(3.1)

**Proof** Let \( C \) be a linear \([n, 3, d]\)-code over \( F_q \).

If \( C \) has repeated columns in its generator matrix, it may be assumed that the first two columns are \((1, 0, 0)^t\). The matrix obtained by deleting the first two columns and the first row generates an \([n - 2, 2, d]\) linear code. Applying the Griesmer bound,

\[
n - 2 \geq d + \left\lfloor \frac{d}{q} \right\rfloor,
\]

from which the bound (3.1) follows since \( d \leq q^2 \).

If \( C \) has no repeated columns, then an \((n, n - d)\)-arc in \( PG(k - 1, q) \) is obtained. By assumption,

\[
n \leq (n - d - 1)q + 1,
\]

and so

\[
n(q - 1) \geq d + d(q - 1) + q - 1.
\]

Now, dividing by \( q - 1 \) gives the bound (3.1) for \( d \leq q^2 \).
Corollary 3.2. Suppose one of the following holds:

(i) The condition \( r \leq (q + 3)/2 \) translates to \( n \leq d + (q + 3)/2 \). Hence either the bound (3.1) holds or \( n \geq d + (q + 5)/2 \), which is a better bound if \( d \leq (q - 1)(q + 3)/2 \).

(ii) The condition \( r < \sqrt{2q} + 1 \) translates to \( n < \sqrt{2q} + 1 + d \). Hence either the bound (3.1) holds or \( n \geq \sqrt{2q} + 1 + d \) which is a better bound if \( d \leq (q - 1)\sqrt{2q} \). If equality in the bound violates the condition \( (n - d, q) = 1 \), then \( [d/(q - 1)] \equiv -1 \pmod{p} \).

(iii) The condition that there is a line skew from the \((n, r)\)-arc translates to the condition that there is a codeword of weight \( n \).

(iv) The condition \( r \mid q \) translates to \( n = d + pe \) for some \( e \). The bound (3.1) holds or \( n \geq d + \frac{1}{4}\sqrt{q} \), which is a better bound if \( d \leq (q - 1)(\frac{1}{4}\sqrt{q} - 1) \).

\[ \square \]

**Corollary 3.2.** Suppose one of the following holds:

(i) \( q \) is prime and \( d \leq q^{k-3}(q - 1)(q + 3)/2 \);

(ii) \( \left[d/((q - 1)q^{k-3})\right] \neq q - 1 \pmod{p} \) and \( d \leq \sqrt{2q}(q - 1)q^{k-3} \);

(iii) \( \left[d/((q - 1)q^{k-3})\right] \neq q - 1 \pmod{p} \), \( d \leq q^{k-1} \) and there is a codeword of weight \( n \);

(iv) \( q \) is odd, \( n = \sum_{i=0}^{k-3} \left[d/q^i\right] + p^e \) for some \( e \) and \( d < (\frac{1}{4}\sqrt{q} - 1)(q - 1)q^{k-3} \).

Then a linear \([n, k, d]\) code over \( \mathbb{F}_q \) satisfies

\[
\sum_{i=0}^{k-3} \left[d/q^i\right] + \left[d/(q - 1)q^{k-3}\right] + \left[d/q^{k-1}\right].
\]

(3.2)

**Proof** Let \( k \geq 4 \) and let \( C \) be a linear \([n, k, d]\)-code over \( \mathbb{F}_q \).

If \( C \) has repeated columns in its generator matrix, assume that the first two columns are \((1, 0, \ldots, 0)^t\). The matrix obtained by deleting the first two columns and the first row generates an \([n - 2, k - 1, d]\) linear code. Applying the Griesmer bound,

\[
n - 2 \geq \sum_{i=0}^{k-2} \left[d/q^i\right],
\]

from which the bound (3.1) follows since \( d \leq q^{k-1} \).

If \( C \) has no repeated columns in its generator matrix, then let \( K \) be the corresponding \((n, n - d)\) arc in \( \text{PG}(k - 1, q) \). There is a hyperplane \( H \) meeting \( K \) in \( n - d \) points or else \( C \) would have minimum distance more than \( d \). Let \( e \) be the minimum such that \( H \cap K \) is an \((n - d, n - d - e)\)-arc in \( \text{PG}(k - 2, q) \) and let \( L \) be a hyperplane meeting \( H \cap K \) in \( n - d - e \) points. Then counting points of \( K \) on hyperplanes containing \( L \) gives

\[
n \leq eq + n - d,
\]

and hence \( e \geq \lceil d/q \rceil \). Thus \( H \cap K \) gives us an \([n - d, k - 1, \lceil d/q \rceil]\) linear code.

By iteration an \([n - d - \lceil d/q \rceil - \ldots - \lceil d/q^{k-4} \rceil, 3, \lceil d/q^{k-3} \rceil]\) linear code is obtained. Now, according to the conditions, Theorem 3.1 can be applied. \( \square \)
4 Large \((n, r)\)-arcs in small planes

Table B is an update of [34, Table 5.4] including results of Daskalov and Medotieva [22, 25] and the many new constructions of Braun et al. [16].

The new \((n, r)\)-arcs from [16] were found in the following way. Let \(M\) be the point-line incidence matrix of PG\((2, q)\). Then an \((n, r)\)-arc is given by a vector \(x \in \{0, 1\}^{q^2 + q + 1}\) with the property that \(x\) has \(n\) coordinates equal to 1 and the coordinates of \(Mx\) are at most \(r\). Even for small \(q\) the computation involved in solving these equations is unfeasible. For this reason the authors of [16] choose a subgroup \(G\) of the automorphism group of PG\((2, q)\) and consider the reduced incidence matrix \(M^G\) whose columns are the \(G\)-orbits on the points and whose rows are the \(G\)-orbits on the lines. If \(t\) is the number of orbits and \(w_j\) is the size of orbit of points corresponding to the \(j\)-th column of \(M^G\), an \((n, r)\)-arc is given by a vector \(x \in \{0, 1\}^t\) with the property that \(\sum_{j=1}^{t} x_j w_j = n\) and the coordinates of \(M^G x\) are at most \(r\). Various small subgroups \(G\) are used, the corresponding systems of diophantine equations are solved exhaustively by computer using lattice-point enumeration.

Table C provides all the references for the lower and upper bounds in Table B. The reference \([\ast]\) means that the bound can be deduced from the bounds in Section 2. In the case of a \((98, 7)\)-arc in PG\((2, 16)\), a \((165, 11)\)-arc in PG\((2, 16)\), a \((182, 12)\)-arc in PG\((2, 16)\), there is equality in the bound of Theorem 2.1. This implies that such arcs, if they have no skew line, have only \(e\)-secants and \(r\)-secants. Easy counting arguments give a contradiction. If they have a skew line the bound from [4] can be used.

The reference \([\ast\ast]\) refers to the following slight improvement of [2, Theorem 4.2]. If there is equality in the bound and \(r \geq (q + 1)/2\) then the number of \(r\)-secants can be counted, following the same arguments as used for the small planes in [5], and is \((q - 1)(2q + 3 - 2r)(q + 1)/2(q + 1 - r)\). If this number is not an integer then the bound can be improved to
\[
 n \leq (r - 1)q + r - (q + 3)/2.
\]
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Table B: The size of the largest \((n, r)\)-arc in \(PG(2, q)\) for small \(q\)

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Table C: References for the size of the largest \((n, r)\)-arc in \(PG(2, q)\) for small \(q\)

5 Open problems

1. In most cases no example is known of an \((n, r)\)-arc with \(n/q\) large, say \(n/q > r - 2\). The best that can be done in general is to take (a) for \(r < q/2\) the union of \(\lfloor r/2 \rfloor\) conics, which
gives $n/q > r/2$ and (b) for $r > q/2$ large the complement of the union of $2(q - r) + 1$ lines of a dual $(2(q - r) + 1, 2)$-arc, which gives $n/q > q - 2r + (2r^2 - r)/q$.

2. In the case $r = 3$ there is a construction of size $q + [2\sqrt{q}]$ for all $q$ and an upper bound of $n \leq 2q + 1$. Any progress on determining a constant $c$ such that the upper bound $n/q < c < 2$ for $q$ large enough, or a construction where $n/q > c > 1$ for infinitely many $q$ will be rewarded by a cheque for 10,000 Hungarian florins from Prof. A. Blokhuis.

3. In the case $r = q - 1$ Braun et al.’s [16] discovery of a $(145, 12)$-arc in $PG(2, 13)$ ends speculation that an $(n, q - 1)$-arc in $PG(2, q)$, $q$ prime, satisfies $n \leq (q - 2)q + 1$; the so-called 3p conjecture for double blocking sets, see [5]. It is known from [5] that $n \leq (q - 2)q + \frac{4q - 3}{2}$ but in general there is no better construction than the complement of three non-concurrent lines, which provides an example with $n = (q - 2)q + 1$. Any construction of a family of $(n, q - 1)$-arcs in $PG(2, q)$, for infinitely many $q$ prime, with $n \geq (q - 2)q + 2$ would be of interest.

4. For $q$ prime, there are upper bounds on $n$ due to Blokhuis which appear in [2]. For $q$ non-prime and $r > \sqrt{q} + 1$, there are few upper bounds on $n$ that use the fact that the projective plane is Desarguesian; in other words only counting arguments are used. The only exceptions are when $r > q - q^{1/6}$ and $q$ is square.

References


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