Partial unitals and related structures in Desarguesian planes

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Abstract

It is shown that a partial unital with more than $q\sqrt{q}+1-\sqrt{q}$ points in the Desarguesian plane of order q can be extended to a unital.

1. Introduction

A $(k,n)-$*arc* in a projective plane Π is a set of k points, at most n on every line. If the order of the plane is q, where q is a square, a *unital* $\mathcal{U}$ is defined to be a $(q\sqrt{q}+1, \sqrt{q}+1)$-arc meeting all lines of Π in 1 or $\sqrt{q}+1$ points. Lines meeting $\mathcal{U}$ in exactly one point are called *tangent* lines and all other lines are *secant* lines. A unital is a minimal *blocking set*; that is it contains at least one point on every line and is minimal since each point lies on a tangent. A minimal blocking set is also called an *irreducible* blocking set by some authors. For a minimal blocking set $\mathcal{B}$ the following upper and lower bounds are due to Bruen [4] and Bruen and Thas [5],

$$q+\sqrt{q}+1 \leq |\mathcal{B}| \leq q\sqrt{q}+1.$$

A *partial unital* is a $(k, \sqrt{q}+1)$-arc $\mathcal{X}$ such that each point of $\mathcal{X}$ lies on a tangent.

A *flag* of Π is an incident point-line pair. A *strong representative system* $\mathcal{S}$ is a set of flags

$$\mathcal{S} = \{(P_1, l_1), ..., (P_s, l_s)\}$$

such that

$$P_i \in l_j \iff i = j.$$

It was shown by Illés, Szőnyi and Wettl [6] that the size s of a strong representative system $\mathcal{S}$ satisfies $s \leq q\sqrt{q}+1$ with equality if and only if $\mathcal{S}$ consists of the incident point-line pairs of a unital. i.e. such a pair consists of a point of the unital and the tangent at that point. Partial unitals also give rise to a strong representative systems. $\mathcal{S}$ is said to be *maximal* if it cannot be extended to a strong representative system of size $s+1$.

The following two results are due to Blokhuis and Metsch and appeared in [3]. Note that neither result requires the condition that Π be Desarguesian, i.e. $PG(2,q)$.

Theorem 1.1 *If $q \geq 49$ is a square, then a strong representative system $\mathcal{S}$ of size $q\sqrt{q}$ is part of a unital. Consequently, there is no minimal blocking set of size $q\sqrt{q}$.*

Theorem 1.2 *If $q \geq 25$, then a partial unital of size $q\sqrt{q}-1$ is either a minimal blocking set or part of a unital.*

In the same paper, Blokhuis and Metsch further improve the previous Theorem when q is odd and the plane is Desarguesian.

Theorem 1.3 *For odd $q \geq 25$, a partial unital of size $q\sqrt{q}-1$ in $PG(2,q)$ is part of a unital.*

In the same paper, the authors pose the following question: If q is even, does there exist a partial unital of size $q\sqrt{q}-1$ in $PG(2,q)$ which meets every line ? i.e. is a blocking set. In this paper it is shown that no such structure exists, in fact we are able to prove the following theorem:

Theorem 1.4 *A partial unital $\mathcal{X}$ in $PG(2,q)$ with*

$$q\sqrt{q}+1-\sqrt{q} < |\mathcal{X}| < q\sqrt{q}+1$$

can be extended to a unital.

The techniques used in this paper are based on those used in the papers by Ball, Blokhuis and O'Keefe [2] concerning unitals and Ball and Blokhuis [1] concerning the incompleteness of maximal arcs. Throughout it is assumed that q is square.

2. Partial unitals and proof of the theorem

Let $\mathcal{X}$ be a partial unital of size $q\sqrt{q}+1-\epsilon$ in $PG(2,q)$ where $\sqrt{q}>\epsilon>0$ and let $\mathcal{L}$ be the set of lines that is the union of the tangents and external lines to $\mathcal{X}$. Let $\mathcal{G}$ be the set of points not in $\mathcal{X}$ which lie on exactly $\sqrt{q}+1$ elements of $\mathcal{L}$. Note that a point not in $\mathcal{X}$ lies on at most $\sqrt{q}+1$ elements of $\mathcal{L}$. If a point lies on $\sqrt{q}+2$ tangents and external lines then

$$q\sqrt{q}+1-\epsilon=|\mathcal{X}|\leq(\sqrt{q}+1)(q-\sqrt{q}-1)+\sqrt{q}+2,$$

which implies $\epsilon\geq\sqrt{q}$. Let $\mathcal{H}$ be the set of points not in $\mathcal{X}\cup\mathcal{G}$. We will show that points in $\mathcal{H}$ lie on at most one line in $\mathcal{L}$. This will lead swiftly to the conclusion that $\mathcal{X}\cup\mathcal{H}$ is a unital.

Since $\epsilon<\sqrt{q}$ and $\mathcal{X}$ is a partial unital there exists exactly one tangent through each point of $\mathcal{X}$ and hence exactly $q\sqrt{q}+1-\epsilon$ tangents in all. Therefore, we have $q^2\sqrt{q}+q-\epsilon q$ pairs (P,T), where P is a point not in $\mathcal{X}$ and T is a tangent.

Points of $\mathcal{G}$ lie on $\sqrt{q}+1$ tangents and at worst all points in $\mathcal{H}$ lie on $\sqrt{q}$ tangents. Now considering each point not in $\mathcal{X}$ lying on $\sqrt{q}$ tangents this still leaves pairs (P,T), which correspond to one extra tangent through a point of $\mathcal{G}$. This then implies that

$$|\mathcal{G}|\geq q^2\sqrt{q}+q-\epsilon q-(q^2+q-q\sqrt{q}+\epsilon)\sqrt{q}\geq q^2-2q\sqrt{q}+q+\sqrt{q}.$$

Let $\mathcal{R}$ be the set of (q^2+q+1)-st roots of unity (alternatively non-zero $(q-1)$-st powers) in $GF(q^3)$. Consider the points of $PG(2,q)$ as the elements of $\mathcal{R}$ and the sets $\mathcal{X}$, $\mathcal{G}$ and $\mathcal{H}$ as subsets of $\mathcal{R}$.

Let $x\in\mathcal{R}$ and write $x=a^{q-1}$. For any $\epsilon=\delta^{q-1}$ we have that

$$x^{q+1}+\epsilon x+\epsilon^{-q}=a^{q^2-1}+\delta^{q-1}a^{q-1}+\delta^{-q^2+q}=a^{-1}\delta^qTr(a\delta^{-q^2})$$

where $Tr\colon GF(q^3)\to GF(q)$ is the trace function $Tr(x)=x^{q^2}+x^q+x$. It follows immediately that the set of elements that are zeros of

$$x^{q+1}+\epsilon x+\epsilon^{-q}$$

correspond to points on a line since any line is given by a plane through the origin when $GF(q^3)$ is interpreted as a 3-dimensional vector space over $GF(q)$.

For x, $y\in\mathcal{R}$ we can calculate that the line joining them is parameterised by $\epsilon=-(x^{q+1}-y^{q+1})/(x-y)$, and so we have the condition that x, y and z are collinear points if and only if

$$\frac{x^{q+1}-y^{q+1}}{x-y}=\frac{x^{q+1}-z^{q+1}}{x-z}.$$

To simplify computations it is often simpler to consider points $1/x$, y and z, and it follows that these are collinear if and only if

$$\frac{1-x^{q+1}y^{q+1}}{1-xy}=\frac{1-x^{q+1}z^{q+1}}{1-xz}.$$

Elements of $GF(q^3)$ of the form $(1-u^{q+1})/(1-u)$, where u is a non-zero $(q-1)$-st power, are roots of the polynomial $1-t^q+t^{q+1}$. There is a one to one correspondence between the $(q+1)$ roots of the polynomial $1-t^q+t^{q+1}$ and the $(q+1)$ directions of lines through a point.

Note that

$$-u(1-u)^{q-1}=-(u-u^{q+1})/(1-u)=1-(1-u^{q+1})/(1-u),$$

and that if $1/w=1-t$ where t satisfies $1-t^q+t^{q+1}=0$ then $1-w^q+w^{q+1}=0$.

Define the polynomials F in two variables and σ_k in one variable by

$$F(t,x):=\prod_{b\in\mathcal{R}\setminus\mathcal{X}}(1+bx(1-bx)^{q-1}t)=\sum_{k=0}^{q^2-q\sqrt{q}+q+\epsilon}\sigma_k t^k,$$

where σ_k is the k-th elementary symmetric function of the set of polynomials $\{bx(1-bx)^{q-1}\mid b\in\mathcal{R}\setminus\mathcal{X}\}$, a polynomial of degree at most kq in x.

For $1/x_0\in\mathcal{X}$ we get that

$$F(t,x_0)=(1-t^q+t^{q+1})^{q-\sqrt{q}}(1+\sigma_{\sqrt{q}}t^{\sqrt{q}})\left(\sum_{j=0}^{\epsilon}\sigma_j t^j\right),$$

since in this case every line passing through the point $1/x_0$ contains at least $q-\sqrt{q}$ points of $PG(2,q)\setminus\mathcal{X}$, so the multiset $\{bx(1-bx_0)^{q-1}\}$ consists of every root of $1-t^q+t^{q+1}$ repeated $q-\sqrt{q}$ times and some factor repeated q times, corresponding to the direction of the tangent at the point $1/x_0$.

For $1/x_0\in\mathcal{G}$ we get that

$$F(t,x_0)=(1-t^q+t^{q+1})^{q-\sqrt{q}-1}\left(\sum_{i=0}^{\sqrt{q}+1}\gamma_{i\sqrt{q}}t^{i\sqrt{q}}\right)\left(\sum_{j=0}^{\epsilon}\sigma_j t^j\right),$$

where $\gamma_{i\sqrt{q}}=\sigma_{i\sqrt{q}}$ for $i<\sqrt{q}$, since in this case $1/x_0$ contains at least $q-\sqrt{q}-1$ other points of $PG(2,q)\setminus\mathcal{X}$, so the multiset $\{bx(1-bx)^{q-1}\}$ consists of every root of $1-t^q+t^{q+1}$ repeated $q-\sqrt{q}-1$ times and $\sqrt{q}+1$ factors repeated $q-1$ times, those corresponding to the directions of the $\sqrt{q}+1$ tangents and external lines at the point $1/x_0$.

We can deduce from the shape of $F(t, x_0)$ for $1/x_0 \in \mathcal{X} \cup \mathcal{G}$ that

$$\sigma_{r\sqrt{q}+j}(x_0) = \sigma_j(x_0)\sigma_{r\sqrt{q}}(x_0)$$

for $r < \sqrt{q}$ and $0 < j \leq \epsilon$ and that

$$\sigma_{r\sqrt{q}+j}(x_0) = 0$$

for $r < \sqrt{q}$ and $\epsilon < j < \sqrt{q}$. Previously we had that

$$|\mathcal{X} \cup \mathcal{G}| \geq q^2 - q\sqrt{q} + q$$

and so these equalities are in fact identities for $r \leq \sqrt{q} - 2$ since the degree of any of the equalities is at most $(r\sqrt{q}+j)q \leq (q-2\sqrt{q}+\sqrt{q}-1)q = q^2-q\sqrt{q}-q$.

Consider $1/x_0 \in \mathcal{H}$ a point on $\sqrt{q}+1-l$ tangents and external lines where $l \leq \sqrt{q} - 2$. Note that $l > 0$ since $1/x_0 \notin \mathcal{G}$. Then

$$F(t, x_0) = (1 - t^q + t^{q+1})^{q-\sqrt{q}-1} \left(\sum_{i=0}^{\sqrt{q}+1-l} \gamma_{i\sqrt{q}} t^{i\sqrt{q}} \right) \left(\sum_{j=0}^{l\sqrt{q}+\epsilon} \delta_j t^j \right).$$

By the previous argument $\sigma_{i\sqrt{q}+k} \equiv 0$ for $i \leq \sqrt{q} - 2$ and $\epsilon < k < \sqrt{q}$ and so it follows that $\delta_{i\sqrt{q}+k} = 0$ for the same values of i and k since $l \leq \sqrt{q} - 2$. Hence, we can write

$$\sum_{j=0}^{l\sqrt{q}+\epsilon} \delta_j t^j = \sum_{i=0}^{l} \sum_{j=0}^{\epsilon} \delta_{i\sqrt{q}+j} t^{i\sqrt{q}+j}.$$

Define $\beta_{i\sqrt{q}}$ in such a way that

$$\left(\sum_{i=0}^{\infty} \beta_{i\sqrt{q}} t^{i\sqrt{q}} \right) \left(\sum_{i=0}^{\sqrt{q}+1-l} \gamma_{i\sqrt{q}} t^{i\sqrt{q}} \right) = 1;$$

then it is clear that

$$\delta_{i\sqrt{q}+j} = \sum_{k=0}^{i} \beta_{(i-k)\sqrt{q}} \sigma_{k\sqrt{q}+j}$$

and

$$\delta_{i\sqrt{q}+j} = \sum_{k=0}^{i} \beta_{(i-k)\sqrt{q}} \sigma_{k\sqrt{q}} \sigma_j.$$

So we have that

$$\sum_{j=0}^{l\sqrt{q}+\epsilon} \delta_j t^j = \sum_{i=0}^{l} \sum_{j=0}^{\epsilon} \sum_{k=0}^{i} \beta_{(i-k)\sqrt{q}} \sigma_{k\sqrt{q}} \sigma_j t^{i\sqrt{q}+j} = \left(\sum_{j=0}^{\epsilon} \sigma_j t^j \right) \left(\sum_{i=0}^{l} \sum_{k=0}^{i} \beta_{(i-k)\sqrt{q}} \sigma_{k\sqrt{q}} t^{i\sqrt{q}} \right).$$

and

$$F(t,x_0)=(1-t^q+t^{q+1})^{q-\sqrt{q}-1}\left(\sum_{i=0}^{\sqrt{q}+1}\phi_{i\sqrt{q}}t^{i\sqrt{q}}\right)\left(\sum_{j=0}^{\epsilon}\sigma_j t^j\right)$$

for some $\phi_{i\sqrt{q}}$. This, however, is not possible since it implies $F(t,x_0)$ has $\sqrt{q}+1$ factors repeated at least $q-1$ times and that the $\sqrt{q}+1$ lines through $1/x_0$ that these factors correspond to have at most one point of $\mathcal{X}$. i.e. $1/x_0$ is not in $\mathcal{H}$ at all, but in $\mathcal{G}$. Hence, $l \geq \sqrt{q}-1$ which implies all points of $\mathcal{H}$ are on at most two tangents or external lines.

If we repeat our counting we find that

$$(\sqrt{q}-1)|\mathcal{G}| \geq q(q\sqrt{q}+1-\epsilon)-2(q^2+q-q\sqrt{q}+\epsilon)$$

which implies $|\mathcal{G}| \geq q^2-q\sqrt{q}-2$ and hence $|\mathcal{G}\cup\mathcal{X}| \geq q^2-1-\epsilon > q^2-\sqrt{q}$ and $|\mathcal{H}| \leq q^2+q+1$. Repeating the above with this new bound on $|\mathcal{G}\cup\mathcal{X}|$ implies

$$\sigma_{r\sqrt{q}+j} \equiv \sigma_j\sigma_{r\sqrt{q}}$$

for $r<\sqrt{q}$ and $0<j\leq\epsilon$ and that

$$\sigma_{r\sqrt{q}+j} \equiv 0$$

for $r<\sqrt{q}$ and $\epsilon<j<\sqrt{q}$. Considering again $1/x_0 \in \mathcal{H}$, a point on $\sqrt{q}+1-l$ tangents will imply that $l \geq \sqrt{q}$. i.e. $1/x_0$ is a point on at most one tangent.

The rest of the proof now follows from simple counting. Let $\mathcal{H}^*$ be the set of points in $\mathcal{H}$ on exactly one tangent or external line. Let $|\mathcal{L}|=q\sqrt{q}+1-\epsilon+\Delta$, where Δ is the number of external lines. Lines in $\mathcal{L}$ meet in points of $\mathcal{G}$ which lie on exactly $\sqrt{q}+1$ of them and so

$$\sqrt{q}(\sqrt{q}+1)|\mathcal{G}|=(q\sqrt{q}+1-\epsilon+\Delta)(q\sqrt{q}-\epsilon+\Delta)$$

and all points in $\mathcal{X}\cup\mathcal{H}^*$ lie on exactly one tangent or external line so

$$|\mathcal{X}|+|\mathcal{H}^*|+(\sqrt{q}+1)|\mathcal{G}|=(q+1)(q\sqrt{q}+1-\epsilon+\Delta).$$

We have that $|\mathcal{X}|+|\mathcal{G}|+|\mathcal{H}^*|+|\mathcal{H}\setminus\mathcal{H}^*|=q^2+q+1$ and hence that

$$(\sqrt{q}+1)|\mathcal{H}\setminus\mathcal{H}^*|=(\Delta-\epsilon)(q\sqrt{q}-q-\sqrt{q}+\Delta-\epsilon).$$

Clearly this implies $\Delta\geq\epsilon$. However, $|\mathcal{H}\setminus\mathcal{H}^*|\leq|\mathcal{H}|$ and from above $|\mathcal{H}|<q-\epsilon-2\sqrt{q}<q-2\sqrt{q}$ and so $\Delta\leq\epsilon+1$.

We can rule out the case $\Delta=\epsilon+1$ as this implies $\sqrt{q}(\sqrt{q}+1)|\mathcal{G}|=(q\sqrt{q}+2)(q\sqrt{q}+1)$ and hence $\sqrt{q}|2$ which is not possible for $\sqrt{q}>2$. Therefore $\Delta=\epsilon$ and $|\mathcal{H}\setminus\mathcal{H}^*|=0$. The lines in $\mathcal{L}$ form a dual unital and the set of tangents to this unital is the set $\mathcal{X} \cup \mathcal{H}$. The tangents to a unital form a dual unital and so $\mathcal{X} \cup \mathcal{H}$ is unital. For $\sqrt{q} = 2$ we can compute that $|\mathcal{G}| = 15$ and that $|\mathcal{X}| < 6$ which contradicts the range of $|\mathcal{X}|$.

Hence we have proved the following theorem and corollary:

Theorem 2.1 *A partial unital $\mathcal{X}$ in $PG(2,q)$ with*

$$q\sqrt{q} + 1 - \sqrt{q} < |\mathcal{X}| < q\sqrt{q} + 1.$$

can be extended to a unital.

Corollary 2.2 *A partial unital $\mathcal{X}$ in $PG(2,q)$ with*

$$q\sqrt{q} + 1 - \sqrt{q} < |\mathcal{X}| < q\sqrt{q} + 1.$$

is not a minimal blocking set and does not give rise to a maximal strong representative system.

References

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